TACTILENet

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Distributed Binary Hypothesis Testing Over Noisy Channels

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Joint work with Deniz Gündüz.

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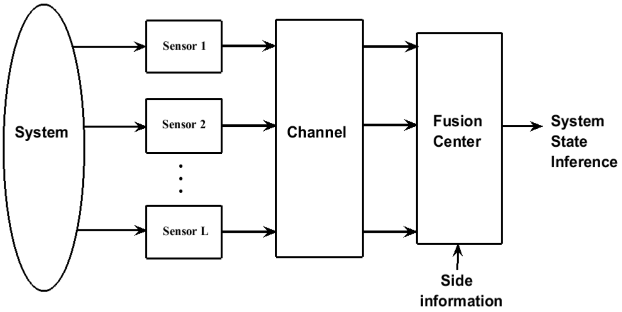
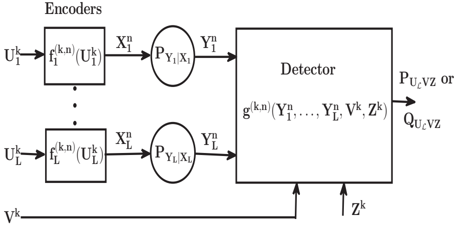
Problem Motivation: Practical scenario



Distributed Binary Hypothesis Testing Over Orthogonal Discrete Memoryless Channels: Model



$$\text{Bandwidth ratio } \tau \triangleq \frac{n}{k}$$

$$H_0 : (U_\mathcal{L}, V, Z) \sim P_{U_\mathcal{L},V,Z}$$
$$H_1 : (U_\mathcal{L}, V, Z) \sim Q_{U_\mathcal{L},V,Z}$$

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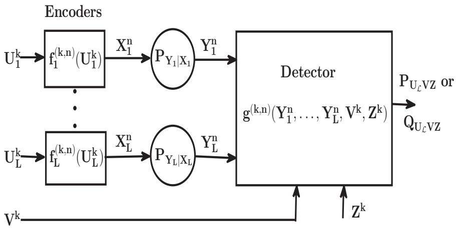

Bandwidth ratio $\tau \triangleq \frac{n}{k}$

$$H_0 : (U_{\mathcal{L}}, V, Z) \sim P_{U_{\mathcal{L}},V,Z}$$
$$H_1 : (U_{\mathcal{L}}, V, Z) \sim Q_{U_{\mathcal{L}},V,Z}$$

Assumption: $P_{U_{\mathcal{L}}Z} = Q_{U_{\mathcal{L}}Z}$ and $P_{VZ} = Q_{VZ}$.

Type 2 error exponent definition

For a given $g^{(k,n)}$ (or decision region $A \subseteq \mathcal{Y}_{\mathcal{L}}^n \times \mathcal{V}^k \times \mathcal{Z}^k$) and encoders $f_1^{(k,n)}, \ldots, f_L^{(k,n)}$,

$$\bar{\alpha}\left(k, n, f_1^{(k,n)}, \ldots, f_L^{(k,n)}, g^{(k,n)}\right) \triangleq P_{Y_{\mathcal{L}}^n V^k Z^k}(A^c)$$

$$\bar{\beta}\left(k, n, f_1^{(k,n)}, \ldots, f_L^{(k,n)}, g^{(k,n)}\right) \triangleq Q_{Y_{\mathcal{L}}^n V^k Z^k}(A)$$

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$$\beta(k, \tau, \epsilon) \triangleq \inf_{\substack{f_1^{(k,n)}, \ldots, f_L^{(k,n)},\ g^{(k,n)} \\ n \leq \tau k}} \left\{ \begin{array}{l} \bar{\beta}\left(k, n, f_1^{(k,n)}, \ldots, f_L^{(k,n)}, g^{(k,n)}, \epsilon\right) s.t. \\ \bar{\alpha}\left(k, n, f_1^{(k,n)}, \ldots, f_L^{(k,n)}, g^{(k,n)}\right) \leq \epsilon \end{array} \right\}$$

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Question: Does a computable characterization exist for the quantity $\lim_{k \to \infty} \frac{-\log(\beta(k,\tau,\epsilon))}{k}$?

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Testing against conditional independence problem (TACI):

$$H_0 : (U_1^k, \ldots, U_L^k, V^k, Z^k) \sim P^k_{U_{\mathcal{L}}VZ}$$
$$H_1 : (U_1^k, \ldots, U_L^k, V^k, Z^k) \sim P^k_{U_{\mathcal{L}}|Z} \times P^k_{V|Z} \times P^k_Z$$
$$\Rightarrow (Y_1^n, \ldots, Y_L^n, V^k, Z^k) \sim P_{Y^n_{\mathcal{L}}|Z^k} \times P^k_{V|Z} \times P^k_Z$$

Previous related works:

1. General hypothesis testing problem with a single observer and rate-limited channel [Ahlswede-Csiszár (1986) and Han (1987)]

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   - With single observer having common and private bit pipes to multiple detectors [Wigger-Timo (2016)]

Bounds for the Type 2 error exponent

Lemma

For any bandwidth ratio $\tau > 0$, we have

(i) $\limsup_{k\to\infty} \frac{\log(\beta(k,\tau,\epsilon))}{k} \leq -\theta(\tau)$, $\forall\ \epsilon \in (0,1)$.

(ii) $\lim_{\epsilon\to 0} \liminf_{k\to\infty} \frac{\log(\beta(k,\tau,\epsilon))}{k} \geq -\theta(\tau)$.

where

$$\theta(\tau) \triangleq \sup_{k} \theta(k,\tau)$$

$$\theta(k,\tau) \triangleq \sup_{\substack{f_1^{(k,n)},\ldots,f_L^{(k,n)} \\ n\leq \tau k}} \frac{D\left(P_{Y_{\mathcal{L}}^n V^k Z^k} || Q_{Y_{\mathcal{L}}^n V^k Z^k}\right)}{k}$$

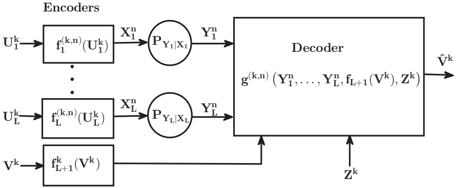
$\theta(\tau)$ for TACI

$$\theta(\tau) = \sup_{\substack{k, f_1^{(k,n)}, \ldots, f_L^{(k,n)} \\ n \leq \tau k}} \frac{D\left(P_{Y_{\mathcal{L}}^n V^k Z^k} || Q_{Y_{\mathcal{L}}^n V^k Z^k}\right)}{k}$$

$$= \sup_{\substack{k, f_1^{(k,n)}, \ldots, f_L^{(k,n)} \\ n \leq \tau k}} \frac{I(Y_{\mathcal{L}}^n; V^k | Z^k)}{k}$$

$$= H(V|Z) - \inf_{\substack{k, f_1^{(k,n)}, \ldots, f_L^{(k,n)} \\ n \leq \tau k}} \frac{H(V^k | Y_{\mathcal{L}}^n, Z^k)}{k}$$

Equivalence between TACI and L-helper JSCC problem:



Goal: To reconstruct V^k losslessly.

What is the minimum rate $R(\tau)$ required at encoder $f_{L+1}^k(\cdot)$ to achieve this?

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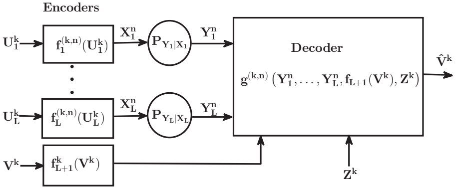

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$$R(\tau) = \inf_{\substack{k,\ f_1^{(k,n)},\ldots,f_L^{(k,n)} \\ n \leq \tau k}} \frac{H(V^k | Y_{\mathcal{L}}^n, Z^k)}{k} \text{ s.t}$$

$$(Z^k, V^k) - U_l^k - X_l^n - Y_l^n,\ l \in \mathcal{L}.$$

$$\theta(\tau) = H(V|Z) - R(\tau)$$

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Lower Bound for $\theta(\tau)$:

Theorem

$$\theta(\tau) \geq H(V|Z) - R^i(\tau) \ \textit{where}$$

$$R^i(\tau) \triangleq \inf_{W_{\mathcal{L}}} \max_{S \subseteq \mathcal{L}} F_S,$$

$$F_S = H(V|W_{S^c}, Z) + I(U_S; W_S|W_{S^c}, V, Z) - \tau \sum_{l \in S} C_l$$

$$(Z,\ V,\ U_{l^c},\ W_{l^c}) - U_l - W_l,\ |\mathcal{W}_l| \leq |\mathcal{U}_l| + 4,\ l \in \mathcal{L} \tag{1}$$

$$I(U_{\mathcal{L}}; W_S|V, W_{S^c}, Z) \leq \tau \sum_{l \in S} C_l. \tag{2}$$

Proof: Source-channel separation theorem for orthogonal MAC + Berger-Tung inner bound.

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Tightness of the bounds for $L = 1$

Lemma

For the TACI problem with $L = 1$ and bandwidth ratio τ,

$$\theta(\tau) = \sup_{W} I(V; W|Z)$$
$$\text{such that } I(U; W|Z) \leq \tau C,$$
$$(Z, V) - U - W, \ |\mathcal{W}| \leq |\mathcal{U}| + 4.$$

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Proof:

1. For $L = 1$, the Markov relation among the r.v.'s in the BT inner and outer bounds match.

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2. Applying the BT inner bound, $R(\tau)$ is the infimum of R' such that

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3. Equivalently,

$$R(\tau) = \inf_{W} \max\big(H(V|W,Z),\ H(V|W,Z) + I(U;W|Z) - \tau C\big), \quad (3)$$
$$\text{such that } I(U;W|V,Z) \leq \tau C.$$
$$\theta(\tau) = H(V|Z) - R(\tau). \quad (4)$$

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Proof (cont):

4. We want to show that $R(\tau) = H(V|W, Z)$ for some W such that $I(U; W|Z) \leq \tau C$.

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5. If W^* achieving the minimum in (3) is such that $I(U; W^*|Z) \leq \tau C$, then $R(\tau) = H(V|W^*, Z)$ and $\theta(\tau) = H(V|Z) - H(V|W^*, Z) = I(V; W^*|Z)$ as required.

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6. On the contrary, suppose that the minimum is achieved for a W^* such that $I(U; W^*|Z) > \tau C$.
   $\Rightarrow R(\tau) = H(V|W^*, Z) + I(U; W^*|Z) - \tau C$ and
   $I(U; W^*|V, Z) \leq \tau C$.

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7. We will show that such a W^* need not be considered.
8. Sufficient to show that $\exists\ \bar{W}$ s.t.,
   (i) $I(U; \bar{W}|Z) = \tau C$
   (ii) $H(V|\bar{W}, Z) + I(U; \bar{W}|Z) - \tau C \leq H(V|W^*, Z) + I(U; W^*|Z) - \tau C$
   (iii) $I(U; \bar{W}|V, Z) \leq \tau C$
   (iv) $(Z, V) - U - \bar{W}$

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Proof (cont):

9. Setting $\bar{W} = W_{p^*}$ suffices, where

$$W_p \triangleq \begin{cases} W^*, & \text{with probability 1-p,} \\ \text{constant,} & \text{with probability p,} \end{cases}$$

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(i) $I(U; W_p|Z)$ and $I(U; W_p|V, Z)$ are decreasing functions of p.

(ii) $H(V|W_p, Z) + I(U; W_p|Z) - \tau C$ is a decreasing function of p.

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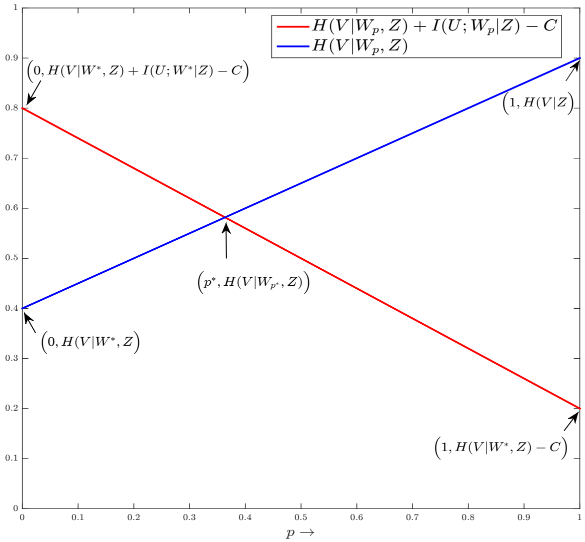

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$$H(V|W_p, Z) = (1-p)H(V|W^*, Z) + pH(V|Z)$$

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By the DPI for $(V, Z) - U - W^*$,

$$\frac{d}{dp}H(V|W_p, Z) = I(V; W^*|Z) \leq I(U; W^*|Z) = \frac{d}{dp}H(U|W_p, Z)$$

Proof (cont):

$$\Rightarrow \frac{d}{dp}\left(H(V|W_p, Z) + I(U; W_p|Z) - \tau C\right)$$
$$= I(V; W^*|Z) - I(U; W^*|Z) \leq 0$$

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Conclusions and Open questions

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Open questions:

- Is the optimal T2EE independent of ϵ?
- Computable characterization of the optimal T2EE for the general hypothesis testing problem.

THANKS FOR THE ATTENTION !